

CONVEX MINORANTS OF RANDOM WALKS AND LÉVY PROCESSES

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ABSTRACT. This article provides an overview of recent work on descriptions and properties of the convex minorant of random walks and Lévy processes as detailed in [1, 15, 16], which summarize and extend the literature on these subjects.

The results surveyed include point process descriptions of the convex minorant of random walks and Lévy processes on a fixed finite interval, up to an independent exponential time, and in the infinite horizon case. These descriptions follow from the invariance of these processes under an adequate path transformation. In the case of Brownian motion, we note how further special properties of this process, including time-inversion, imply a sequential description for the convex minorant of the Brownian meander.

1. INTRODUCTION

The *greatest convex minorant* (or simply convex minorant for short) of a real-valued function $(x_u, u \in U)$ with domain U contained in the real line is the maximal convex function $(\underline{c}_u, u \in I)$ defined on a closed interval I containing U with $\underline{c}_u \leq x_u$ for all $u \in U$. A number of authors have provided descriptions of certain features of the convex minorant for various stochastic processes such as random walks [1], Brownian motion [6, 7, 10, 13, 19, 3], Cauchy processes [4], Markov Processes [11], and Lévy processes (Chapter XI of [12]). Figure 1 illustrates an instance of the convex minorant for each of a random walk, a Brownian motion, and a Cauchy process on a finite interval.

The recent articles [1, 15, 16] provide a relatively complete description of the convex minorant of random walks, Brownian Motion, and Lévy processes which not only encompass much of the somewhat ad hoc previous work on convex minorants, but also provide new tools to derive previously unknown properties of such convex minorants. These three articles together run well over 100 pages and so the purpose of this note is to provide an overview of these works. To this end, we will focus on stating results in a streamlined fashion, referring to [1, 15, 16] where needed to furnish details and proofs.

The layout of the paper is as follows. In Section 2 we discuss the convex minorant of a random walk following [1], and in Section 3 we describe the limiting case of the results of Section 2 - the convex minorant of a Lévy process studied in [16]. In section 4 we provide an overview of additional results in the special case of Brownian motion, found in [15]. We conclude in section 5 by presenting a selection of open problems.

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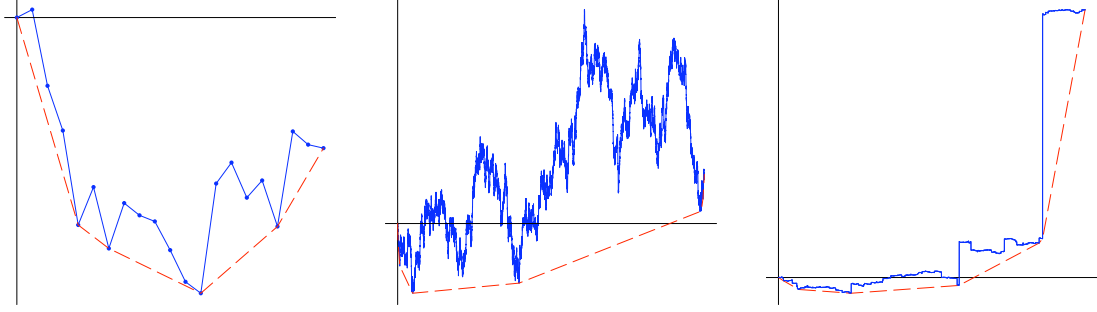


FIGURE 1. Illustration of the convex minorant of a random walk, a Brownian motion, and a Cauchy process on a finite interval.

2. RANDOM WALKS

Let $S_0 = 0$ and $S_j = \sum_{i=1}^j X_i$ for $1 \leq j \leq n$, where X_1, \dots, X_n are exchangeable random variables such that almost surely no two subsets of X_1, \dots, X_n have the same arithmetic mean (satisfied for example if the X_i are i.i.d. with continuous distribution). Let $S^{[0,n]} := \{(j, S_j) : 0 \leq j \leq n\}$, so that $S^{[0,n]}$ is the random walk of length n with increments distributed as X_1, \dots, X_n . As Figure 1 indicates, the convex minorant of $S^{[0,n]}$ consists of piecewise linear segments which we refer to as ‘faces.’ Let F_n be the number of faces of $\underline{C}^{[0,n]}$, the convex minorant of $S^{[0,n]}$, and define

$$0 < N_{n,1} < N_{n,1} + N_{n,2} < \dots < N_{n,1} + \dots + N_{n,F_n} = n$$

be the successive indices j with $0 \leq j \leq n$ such that $S_j = \underline{C}_j$; we refer to $N_{n,i}$ as the ‘length’ of the i th face of $\underline{C}^{[0,n]}$. Finally, let $L_{n,1}, \dots, L_{n,F_n}$ be the lengths of the faces of $\underline{C}^{[0,n]}$ arranged in non-decreasing order. We refer to this sequence as the *partition* of n generated by the convex minorant of $S^{[0,n]}$. Recall the following classical result.

Theorem 1 ([2, 5, 9, 18]). *The sequence $L_{n,1}, \dots, L_{n,F_n}$ of ranked lengths of faces of the convex minorant of $S^{[0,n]}$, a random walk with exchangeable increments with almost surely no subset average ties has the same distribution as the ranked cycle lengths of a uniformly chosen permutation of n elements:*

$$\mathbb{P}(F_n = k, L_i = n_i, 1 \leq i \leq k) = \prod_{j=1}^n \frac{1}{j^{a_j} a_j!}$$

where $a_j := \#\{i : n_i = j\}$, and $n_1 \geq \dots \geq n_k$ with $n_1 + \dots + n_k = n$.

The following natural question was the starting point of our study of convex minorants.

Given the partition of n generated by the faces of the convex minorant of $S^{[0,n]}$, how are the lengths ordered to form the *composition* of n generated by the convex minorant of $S^{[0,n]}$?

In the notation above, the sequence of variables $(N_{n,1}, \dots, N_{n,F_n})$ is the composition of n generated by the convex minorant.

In the case that the X_i are i.i.d. the answer to this question is easy to describe. For $j = 1, \dots, n$ each face of length j is assigned an increment distributed as S_j , independently of all other

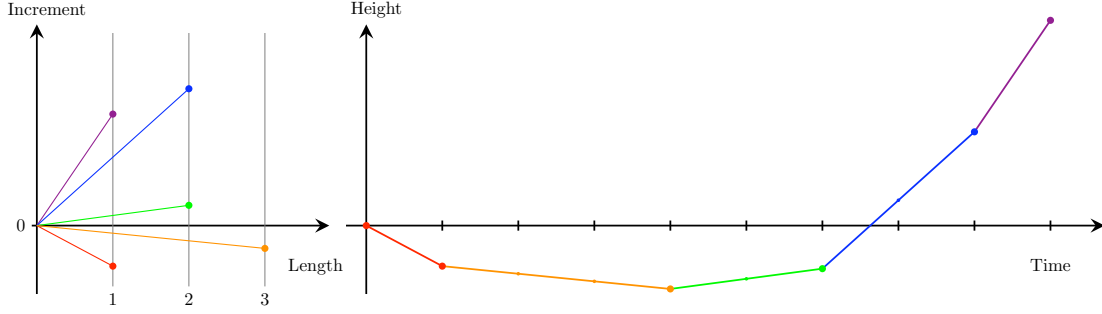


FIGURE 2. Illustration of the convex minorant of a random walk as a point process.

increments, and then the faces are ordered according to increasing slope. Formally, we have the following result.

Theorem 2 ([1]). *Let $(N_{n,1}, \dots, N_{n,F_n})$ be the composition of n induced by the lengths of the faces of the convex minorant of $S^{[0,n]}$. Assuming no subset average ties, the joint distribution of $N_{n,1}, \dots, N_{n,F_n}$ is given by the formula*

$$\mathbb{P}(F_n = k, N_{n,i} = n_i, 1 \leq i \leq k) = \mathbb{P}\left(\frac{S_{n_1}^{(1)}}{n_1} < \frac{S_{n_2}^{(2)}}{n_2} < \dots < \frac{S_{n_k}^{(k)}}{n_k}\right) \prod_{i=1}^k \frac{1}{n_i}$$

for all n_1, \dots, n_k with $n_1 + \dots + n_k = n$, and where for $1 \leq i \leq k$

$$S_{n_i}^{(i)} := S_{n_1 + \dots + n_i} - S_{n_1 + \dots + n_{i-1}} \stackrel{d}{=} S_{n_i}.$$

In particular, if the X_i are independent, then so are the $S_{n_j}^{(i)}$ for $1 \leq i \leq k$.

The special case of Cauchy increments gives rise to the following appealing version of Theorem 2.

Corollary 3. *Suppose that the X_i are independent and such that S_k/k has the same distribution for every k , as when the X_i have a Cauchy distribution. Then*

$$\mathbb{P}(F_n = k; N_{n,i} = n_i, 1 \leq i \leq k) = \frac{1}{k!} \prod_{i=1}^k \frac{1}{n_i},$$

and hence $\{N_{n,i} : 1 \leq i \leq F_n\}$ has the same distribution as the composition of n created by first choosing a random permutation of n and then putting the cycle lengths in uniform random order.

Note that the continuum limit of this result can be read from Bertoin's work [4] and follows from the description provided in [16] as discussed below.

In order to proceed further, it is crucial that we introduce the representation of the convex minorant as a point process of lengths and increments of the faces, where the lengths are chosen according to the cycle structure of a random permutation of n elements and the increments are chosen according to Theorem 2 (independently if the X_i are). Figure 2 illustrates this representation.

From this point, we can use Theorem 2 to provide a construction of the convex minorant of a random walk of a random length in the case of independent increments. We already have some

description in this case since we have a construction conditional on the length, but more can be said. The work of Shepp and Lloyd [17] on the cycle structure of permutations combined with the forthcoming Proposition 9 yield the following result.

Theorem 4 ([1]). *Let $n(q)$ be a geometric random variable with parameter $1 - q$; that is $\mathbb{P}(n(q) \geq n) = q^n$, $n = 0, 1, \dots$. If X_1, X_2, \dots are independent with common continuous distribution, then the point process of lengths and increments of faces the convex minorant of $S^{[0, n(q)]}$ is a Poisson point process on $\{1, 2, \dots\} \times \mathbb{R}$ with intensity*

$$j^{-1}q^j\mathbb{P}(S_j \in dx), \quad j = 1, 2, \dots, \quad x \in \mathbb{R}.$$

Moreover, let $T_i = \sum_{l=1}^i N_{n(q), l}$, $0 \leq i \leq F_{n(q)}$, be the consecutive indices at which $S^{[0, n(q)]}$ meets its convex minorant, so that $T_0 = 0$ and $T_{F_{n(q)}} = n(q)$. Then the sequence of path segments

$$\{(S_{T_i+k} - S_{T_i}, 0 \leq k \leq N_{n(q), i+1}), i = 0, \dots, F_{n(q)} - 1\},$$

is a list of the points of a Poisson point process in the space of finite random walk segments

$$\{(s_1, \dots, s_j) \text{ for some } j = 1, 2, \dots\}$$

whose intensity measure on paths of length j is $q^j j^{-1}$ times the conditional distribution of the path (S_1, \dots, S_j) given that $S_k > (k/j)S_j$ for all $1 \leq k \leq j - 1$.

An important facet of the Poisson point process description is that it provides a decomposition of a random walk up to the index of its minimum. For example, the description of Theorem 4 is a more complete description of the convex minorant of a random walk which was the basis for Spitzer's combinatorial identity [18].

Theorem 5 ([18]). *Let X_1, X_2, \dots be independent with common continuous distribution, $S_0 = 0$, $S_k = \sum_{i=1}^k X_i$ for $k \geq 1$, and $M_n := \min_{0 \leq k \leq n} S_k$. Then*

$$\sum_{n=0}^{\infty} q^n \mathbb{E} e^{itM_n} = \exp \left(\sum_{k=1}^{\infty} \frac{q^k}{k} \mathbb{E} e^{itS_k^-} \right),$$

where $S_k^- = \min\{S_k, 0\}$.

Now, by letting q tend to one in Theorem 4, we obtain a description of the convex minorant of $S^{[0, \infty)}$, a random walk on $[0, \infty)$.

Theorem 6 ([1]). *If X_1, X_2, \dots are independent with common continuous distribution with $\mathbb{E}X_1 \in (-\infty, \infty]$, then the point process of lengths and increments of faces the convex minorant of $S^{[0, \infty)}$ is a Poisson point process on $\{1, 2, \dots\} \times \mathbb{R}$ with intensity*

$$j^{-1}\mathbb{P}(S_j \in dx), \quad j = 1, 2, \dots, \quad \frac{x}{j} < \mathbb{E}X_1.$$

Similar to Theorem 4, there is a companion path space statement which we omit for the sake of brevity.

The key to the results above is a certain property of a transformation of the walk $S^{[0, n]}$, which not only yields the results above, but also provides a construction of the walk jointly with its convex minorant. We will call this transformation the ‘3214’ transformation, as it is described by first dividing the walk $S^{[0, n]}$ into four consecutive paths, and then reordering these four pieces with the third one first, the second one second, and so on.

The ‘3214’ transform of $S^{[0, n]}$ is generated by a random variable U which is uniform on $\{1, \dots, n\}$ and is independent of $S^{[0, n]}$. Given $U = u$, we then define g and d as the indices of the left and

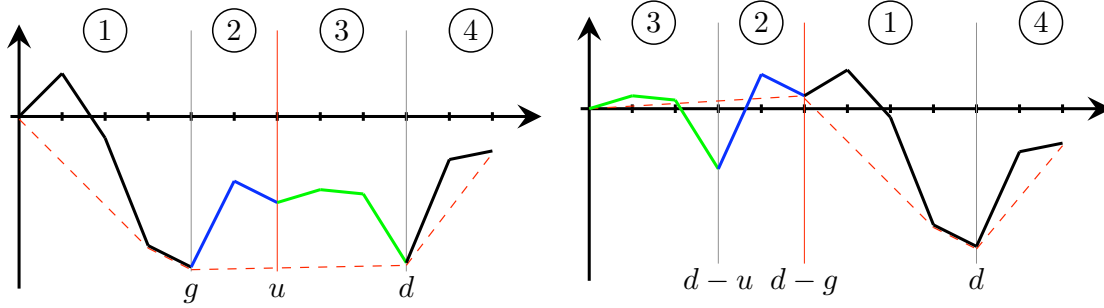


FIGURE 3. Notation and application of the ‘3214’ transformation.

right endpoints of the face of the convex minorant of $S^{[0,n]}$ straddling the index u . Note that g and d are almost surely well defined by this description due to the no subset average ties assumption. Consider the four paths of the random walk on the intervals $[0, g]$, $[g, u]$, $[u, d]$, and $[d, n]$. With this setup, the ‘3214’ transform is defined by reordering the four path fragments of $S^{[0,n]}$ described above to form a new walk path $S_U^{[0,n]}$ in the order 3 – 2 – 1 – 4. Figure 3 illustrates the notation and provides an example of the transformation.

The following lemma summarizes the crucial feature of this transform.

Lemma 7 ([1]). *Let X_1, \dots, X_n be exchangeable random variables with no subset average ties and $S^{[0,n]}$ the random walk generated by the X_i . Let U uniform on $\{1, \dots, n\}$, independent of the X_i and $S_U^{[0,n]}$ the ‘3214’ transform of $S^{[0,n]}$ generated by U . If g and d are the indices of the endpoints of the face of the convex minorant of $S^{[0,n]}$ to the left and right of U , then*

$$(U, S^{[0,n]}) \stackrel{d}{=} (d - g, S_U^{[0,n]}).$$

To see how Lemma 7 corroborates the story above, we introduce discrete *uniform stick breaking* on $[0, n]$, one of the many well-known representations of the distribution of the cycle lengths of a uniformly chosen permutation on n elements.

Definition 8. For an integer n , define the discrete uniform stick breaking sequence of random variables $M_{n,1}, \dots, M_{n,K_n}$ as follows.

- $M_{n,1}$ is uniform on $\{1, \dots, n\}$.
- For $i \geq 1$, if $\sum_{j=1}^i M_{n,j} < n$, then $M_{n,i+1}$ is uniform on $\{1, n - \sum_{j=1}^i M_{n,j}\}$.
- For $i \geq 1$, if $\sum_{j=1}^i M_{n,j} = n$, then set $K_n = i$, and end the process.

We refer to the variables $L_{n,1}, \dots, L_{n,K_n}$ defined to be $M_{n,1}, \dots, M_{n,K_n}$ rearranged in non-increasing order as the *partition* of n generated by uniform stick breaking.

To be explicit, we state the following well-known proposition (see [14]).

Proposition 9. *The partition of n generated by uniform stick breaking has the same distribution as the ranked cycle lengths of a uniformly chosen permutation of n elements.*

From this point, some consideration yields the following implications of Lemma 7 for a walk with i.i.d. increments and no subset average ties:

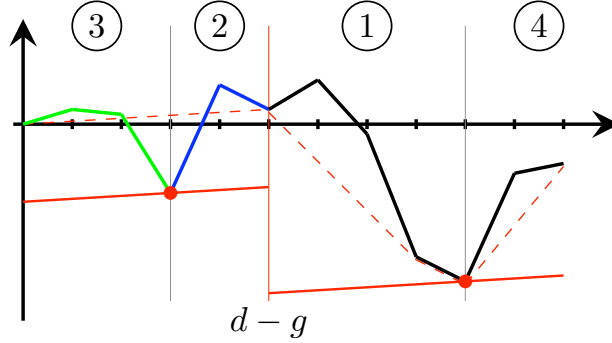


FIGURE 4. Illustration of the inverse ‘3214’ transform for the walk of Figure 3.

- The lengths of the faces of the convex minorant of $S^{[0,n]}$ are distributed as discrete stick breaking.
- Conditional on the lengths of the faces of the convex minorant of $S^{[0,n]}$, the excursions above the segments are independent.
- Given a segment of length j , the excursion above the segment can be realized as the unique cyclic permutation of a random walk of length j equal in distribution to $S^{[0,j]}$ which yields a convex minorant of exactly one segment.

The last item is similar in spirit to Vervaat’s transform of a Brownian bridge to an excursion [21]. As this transformation is not well developed for random walks and Lévy processes in general (some statements for Lévy processes are found in [20]), this last item carries real content.

The proof of Lemma 7 essentially follows from two observations. The first is that given the values of the increments $X_1 = x_1, \dots, X_n = x_n$, S_j is distributed as $\sum_{i=1}^j x_{\sigma_i}$ for $j = 1, \dots, n$ and where σ is a permutation chosen uniformly at random. From this point we only need to show that the ‘3214’ transformation is a bijection between $\{1, \dots, n\} \times$ ‘paths generated from permutations of x_1, \dots, x_n ’ for fixed increments x_i having no subset average ties. The bijection is easily verified after noting that for a given value of $d - g$, the indices at which Segment 1 meets Segment 4 and Segment 3 meets Segment 2 are found by raising a line with slope equal to the mean of the first $d - g$ increments. Figure 4 illustrates this inverse transformation.

If we remove the assumption that almost surely, no two subsets of the X_i have the same mean, then the process of generating excursions described above may generate excursions that meet the corresponding face of the convex minorant at points other than the end points, and excursions that have the same slope. This implies that there is not necessarily a unique cyclic permutation transforming a walk into an excursion, and neither is there necessarily a unique ordering of the excursions that puts them in non-decreasing order of slope. Such technical issues can be dealt with in a straightforward manner, but the details, found in [1], are a little laborious.

3. LÉVY PROCESSES

A real valued process X is a Lévy process on $[0, \infty)$ if $X_0 = 0$, X is cadlag (right continuous with left limits), and X has independent and stationary increments. As is well known, Lévy processes are the continuous scaling limits of discrete time random walks generated by i.i.d. increments, so it is not surprising that continuous analogs of the results of Section 2 hold for Lévy processes. However,

there are a few interesting wrinkles not present in the discrete case and many technical details to be considered in pushing the discrete results to the limit. We restrict our analysis to the case that X_t has continuous distribution for all $t > 0$, which is equivalent to the assumption that X is not a compound Poisson process with drift.

In analogy to the case of random walk, we can view the intervals that a Lévy process is strictly greater than its convex minorant on $[0, 1]$ as an interval partition of the unit interval. The formal statement of this last fact is proved in [16], but also intersects with the work [11].

Proposition 10 ([16]). *Let X be a Lévy process with continuous distribution and \underline{C} the convex minorant of X on $[0, 1]$. Let $\mathcal{O} = \{s \in (0, t) : \underline{C}_s < X_s \wedge X_{s-}\}$ and \mathcal{I} be the set of connected components of \mathcal{O} . The following conditions hold almost surely:*

- (1) *The open set $\mathcal{O} = \{s \in (0, t) : \underline{C}_s < X_s \wedge X_{s-}\}$ has Lebesgue measure 1.*
- (2) *\mathcal{I} is a set of disjoint intervals and the closure of its union is $[0, 1]$.*
- (3) *If (g_1, d_1) and (g_2, d_2) are distinct intervals of \mathcal{I} , then the slopes of \underline{C} over those intervals differ:*

$$\frac{\underline{C}_{d_1} - \underline{C}_{g_1}}{d_1 - g_1} \neq \frac{\underline{C}_{d_2} - \underline{C}_{g_2}}{d_2 - g_2}.$$

For each $(g, d) \in \mathcal{I}$, we refer to g and d as *vertices*, the *length* is $d - g$, the *increment* is $\underline{C}_d - \underline{C}_g$, and the *slope* is $(\underline{C}_d - \underline{C}_g)/(d - g)$.

Because the partition of n generated by the convex minorant of an i.i.d. generated random walk of n steps is distributed as the partition of n generated by the cycles of a random permutation for any increment distribution, we might hope that a similar universal result holds for Lévy processes and also that this universal result might be a limiting continuous distribution of the cycle structure of a random permutation. This is indeed the case, but before stating our result we define this continuous limit.

Definition 11. Define the continuous uniform stick breaking sequence of random variables as the sequence L_1, L_2, \dots defined as follows.

- L_1 is uniform on $[0, 1]$.
- For $i \geq 1$, L_{i+1} is uniform on $[0, 1 - \sum_{j=1}^i L_j]$.

We refer to the variables L_1, L_2, \dots , rearranged in non-increasing order as the *partition* of $[0, 1]$ generated by uniform stick breaking.

The variables L_1, L_2, \dots almost surely sum to one and their law once arranged in decreasing order is referred to as the *Poisson-Dirichlet distribution with parameter one* which is the limiting distribution of the cycle structure of a permutation chosen uniformly at random (see [14]). We can now state the following result and note that a proof in the special case of Brownian motion was sketched in [19].

Theorem 12 ([16]). *The sequence of ranked lengths of faces of the convex minorant of a Lévy process with continuous distributions has the Poisson-Dirichlet distribution with parameter one.*

In light of Theorem 12, the following natural question arises. Given the interval partition of $[0, 1]$ generated by the convex minorant of a Lévy process X with continuous distribution, how are the intervals ordered to form the interval *composition* of $[0, 1]$ generated by the convex minorant of X ?

In total analogy with the answer for i.i.d. random walks, the answer to this question is easy to describe. Given the interval of length l , the increment is distributed as X_l independent of all other increments, and the faces are ordered according to increasing slope.

Theorem 13 ([16]). *Let X be a Lévy process with continuous distribution and let $\{(g_i, d_i)\}_{i \geq 1}$ denote the intervals of \underline{C} , the convex minorant of X on $[0, 1]$. Let L_1, L_2, \dots be generated by uniform stick breaking, $S_0 := 0$, and for $i \geq 1$, define $S_i := \sum_{j=1}^i L_j$. Then we have the following equality in distribution between sequences:*

$$\left((d_i - g_i, \underline{C}_{d_i} - \underline{C}_{g_i}), i \geq 1\right) \stackrel{d}{=} \left((L_i, X_{S_i} - X_{S_{i-1}}), i \geq 1\right).$$

We note here that applying the theorem to a Cauchy process yields the main result of Bertoin [4] and also shows the composition generated by the convex minorant on $[0, 1]$ is a uniform ordering of the generated partition; c.f. Corollary 3.

We can also consider the convex minorant of a Lévy process X on an interval of a random exponential length independent of X to obtain the following analog of Theorem 4.

Theorem 14 ([16]). *Let T be a rate θ exponential random variable, X a Lévy process with continuous distribution which is independent of T and let \underline{C}^T denote the convex minorant of X on $[0, T]$. The point process*

$$\{(d_i - g_i, \underline{C}_{d_i}^T - \underline{C}_{g_i}^T), i \geq 1\}$$

generated by the lengths t and increments x of \underline{C}^T has the same distribution as a Poisson point process with intensity measure

$$\mu(dt, dx) = \frac{e^{-\theta t}}{t} dt \mathbb{P}(X_t \in dx).$$

By integrating out the independent exponential variable, we can also use Theorem 14 gain insight into the structure of the convex minorant of a Lévy process on $[0, 1]$.

For example, this program yields the following neat dichotomy for stable Lévy processes.

Proposition 15 ([16]). *Let X be a symmetric stable process with parameter α ; that is $\mathbb{E}e^{iuX_t} = e^{-t|u|^\alpha}$, and let \mathcal{S} be the set of slopes and \mathcal{T} be the set of times of vertices of the convex minorant of X on $[0, 1]$.*

- *If $1 < \alpha \leq 2$, then \mathcal{S} has no accumulation points, $\mathcal{S} \cap (a, \infty)$ and $\mathcal{S} \cap (-\infty, -a)$ are infinite for $a > 0$, and \mathcal{T} has accumulation points at zero and one only.*
- *If $0 < \alpha \leq 1$, then \mathcal{S} is dense \mathbb{R} and every point of \mathcal{T} is an accumulation point.*

By letting θ tend to zero in Theorem 14, we obtain a description of the convex minorant of X on $[0, \infty)$ which was also derived in [12].

Theorem 16 ([12, 16]). *If X is a Lévy process with continuous distribution and and*

$$I := \liminf_{t \rightarrow \infty} \frac{X_t}{t} \in (-\infty, \infty],$$

then the lengths t and increments x of the convex minorant of X on $[0, \infty)$ is a Poisson point process with intensity

$$\frac{\mathbb{P}(X_t \in dx)}{t}, \quad x < It.$$

Both of the previous theorems carry an Itô type excursion theory analogous to that of Theorem 4 for random walks, see [16, Thm. 4].

Theorems 12 and 13 follow from a direct analog of Lemma 7 for a ‘3214’ transform for Lévy processes. The proof uses limiting arguments which crucially hinge on certain regularity conditions for Lévy processes governing the behavior of the process at the vertices of the convex minorant.

4. BROWNIAN MOTION

Since Brownian motion is a Lévy process (and stable with index 2), the results of the previous section apply to the convex minorant of Brownian motion, and some of these results were known (from [6, 7, 10, 13, 19]). However, Brownian motion offers extra analysis due to its special properties among Lévy processes (e.g. continuity and time inversion). We begin by noting the following special case of Theorem 14.

Theorem 17 ([10]). *Let Γ_1 be an exponential random variable with rate one. The lengths x and slopes s of the faces of the convex minorant of a Brownian motion on $[0, \Gamma_1]$ form a Poisson point process on $\mathbb{R}^+ \times \mathbb{R}$ with intensity measure*

$$\frac{\exp\{-\frac{x}{2}(2+s^2)\}}{\sqrt{2\pi x}} ds dx, \quad x \geq 0, s \in \mathbb{R}.$$

As with random walks and Lévy processes, the minimum on $[0, T]$ of a Brownian motion is a distinguished point of the convex minorant and the process after the minimum can be described by restricting the point process of slopes and increments to those points with positive slopes. Due to Proposition 15, we can define

$$\alpha_0 < \alpha_1 < \alpha_2 < \dots < 1$$

with $\alpha_n \uparrow 1$ as $n \rightarrow \infty$ to be times of vertices of the convex minorant of a Brownian motion B on $[0, 1]$, arranged relative to

$$\alpha_0 := \operatorname{argmin}_{0 \leq t \leq 1} B_t.$$

Brownian scaling and Theorem 17 yield an implicit description of the distribution of the sequence $(\alpha_i)_{i \geq 0}$. Moreover, Denisov's decomposition for Brownian motion at the minimum [8] implies that the process after the minimum is a Brownian meander, for which we now provide an alternate description. First we make the following definition.

Definition 18. We say that a sequence of random variables $(\tau_n, \rho_n)_{n \geq 0}$ satisfies the (τ, ρ) recursion if for all $n \geq 0$:

$$\rho_{n+1} = U_n \rho_n \quad \text{and} \quad \tau_{n+1} = \frac{\tau_n \rho_{n+1}^2}{\tau_n Z_{n+1}^2 + \rho_{n+1}^2}$$

for the two independent sequences of i.i.d. uniform $(0, 1)$ variables U_n and i.i.d. squares of standard normal random variables Z_n^2 , both independent of (τ_0, ρ_0) .

Theorem 19 ([15]). *Let $(X(v), 0 \leq v \leq t)$ be a Brownian meander of length t , and let $(\underline{C}(v), 0 \leq v \leq t)$ be its convex minorant. The vertices of $(\underline{C}(v), 0 \leq v \leq t)$ occur at times $0 = V_0 < V_1 < V_2 < \dots$ with $\lim_n V_n = t$. Let $\tau_n := t - V_n$ so $\tau_0 = t > \tau_1 > \tau_2 > \dots$ with $\lim_n \tau_n = 0$. Let $\rho_0 = X(t)$ and for $n \geq 1$ let $\rho_0 - \rho_n$ denote the intercept at time t of the line extending the segment of the convex minorant of X on the interval (V_{n-1}, V_n) . The convex minorant \underline{C} of X is uniquely determined by the sequence of pairs (τ_n, ρ_n) for $n = 1, 2, \dots$ which satisfies the (τ, ρ) recursion with*

$$\rho_0 \stackrel{d}{=} \sqrt{2t\Gamma_1} \quad \text{and} \quad \tau_0 = t,$$

where Γ_1 is an exponential random variable with rate one.

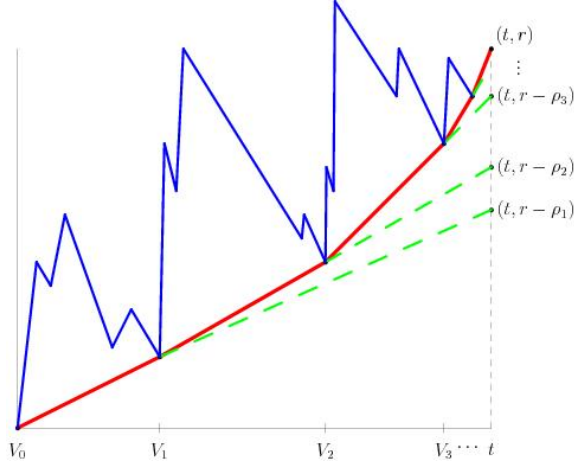


FIGURE 5. An illustration of the notation of Theorem 19. The blue line represents a Brownian meander of length t , and the red line its convex minorant. Note also that $V_i := t - \tau_i$ for $i = 0, 1, \dots$

In [15], we use the descriptions provided by Theorems 17 and 19 (and the interplay between them as per Denisov's decomposition [8]) to derive various properties about the convex minorant of Brownian motion on $[0, 1]$, such as formulas for densities of the α_i . We also use the equivalence of the two descriptions to discover new identities between related quantities in each description. We conclude this section with an elementary example of such an identity; we leave a direct proof as a challenge to the reader.

Corollary 20 ([15]). *Let W and Z be standard normal random variables, U uniform on $(0, 1)$, and R Rayleigh distributed having density $re^{-r^2/2}$, $r > 0$. If all of these variables are independent, then*

$$\left(\frac{W^2 + (1-U)^2 R^2}{1 + U^2 R^2 / Z^2}, \frac{(1-U)R}{\sqrt{T}} \right) \stackrel{d}{=} \left(Z^2, \frac{(1-U)R}{\sqrt{T}} \right).$$

Note that the two coordinate variables on the right are independent.

5. OPEN PROBLEMS

We end this note with a list of open problems.

- Under what conditions is the right derivative of the convex minorant of a Lévy process with continuous distribution discrete, continuous, or mixed?
- Provide a description of the convex minorant of a continuous time process with *exchangeable* increments.
- Provide a framework independent of the convex minorant of Brownian motion that explains the equivalence of the Poisson point process of Theorem 17 with the sequential description of Theorem 19.
- Is there a version of the sequential description of Theorem 19 for random walks or Lévy processes?

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